Journal of Applied Mechanics and Technical Physics, Vol. 38, No. 3, 1997

HOMOGENEOUS DEFORMATION OF A CONTINUOUS MEDIUM

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UDC 539.37

Most mathematical models used in continuum mechanics are of a phenomenological nature. This means that they are based on experimental data on deformation of one or another material. The question arises: What should these experiments be? In constructing a model, one can, in principle, use any experiments; for example, for solids, these are experiments on indentation of various kinds of punches. However, to interpret them, one has to advance a continuum model, to solve a boundary-value problem within the framework of this model, to compare the result with the experiment, to correct the model, etc. As a rule, considerable difficulties arise at the stage of solution of a boundary-value problem. Experiments for which calculations are simplified are more preferable, for example, experiments on torsion of thin-walled tubular specimens. And ideal are experiments whose interpretation does not require the solution of a boundary-value problem altogether. All stresses and strains are then determined immediately using the known boundary displacements and forces, whatever the rheology of the medium.

The simplest example is all-around compression of a body by a pressure p. Let the body be shaped as a ball with radius R, the ball center be immobile, and $u(R)$ be the radial displacement of the boundary. With some general restrictions, one can assert that the stress-tensor components are $\sigma_{ij} = p\delta_{ij}$ (δ_{ij} is the Kronecker symbol), and the displacements are $u(r) = u(R)r/R$ (r is the distance from the center). Of course, using only all-around compression experiments is not sufficient. It is necessary to study more complicated loading programs. The problem therefore arises to describe all theoretically ideal methods of loading which could be applied to develop mathematical models of various media and also to find the parameters of these models. Let us pass to a more rigorous statement which allows us to reveal additional restrictions.

1. Formulation of the Problem. In the general case, the deformation of a continuous medium is described by the closed system

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho X_i = \rho \frac{dv_i}{dt}; \tag{1.1}$$

$$\sigma_{ij} = R_{ij}[e_{kl}, \varepsilon_{kl}], \qquad \varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\right),$$
$$2e_{kl} = \frac{\partial u_k}{\partial x_l^0} + \frac{\partial u_l}{\partial x_k^0} + \frac{\partial u_1}{\partial x_k^0}\frac{\partial u_1}{\partial x_l^0} + \frac{\partial u_2}{\partial x_k^0}\frac{\partial u_2}{\partial x_l^0} + \frac{\partial u_3}{\partial x_k^0}\frac{\partial u_3}{\partial x_l^0}; \tag{1.2}$$

$$u_i = f_i(t_1, x_1, x_2, x_3) \quad \text{for} \quad (x_1, x_2, x_3) \in S_t; \tag{1.3}$$

$$u_i\Big|_{t=0} = g_i(x_1^0, x_2^0, x_3^0), \quad v_i\Big|_{t=0} = G_i(x_1^0, x_2^0, x_3^0). \tag{1.4}$$

Here σ_{ij}, e_{kl}, and ε_{kl} are the components of the stress tensor, strains, and strain rates; u_i, v_i, and ρX_i are the components of the displacement vectors, velocities, and mass forces; ρ is the density; S_t is the boundary of the region at moment t; f_i, g_i, and G_i are the boundary and initial conditions, x_i are the Cartesian coordinates; x_i^0 are the coordinates of a material point at the initial moment of time. All subscripts take on the values 1, 2, and 3. The constitutive equations (1.2) are written symbolically via the functionals R_{ij}.

System (1.1)–(1.4) is closed. Hence, if the data on at least one of them are not available, then the problem becomes indeterminate. Exceptional situations can, however, occur when information only on the

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Mining Institute, Siberian Division, Russian Academy of Sciences, Novosibirsk 630091. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 38, No. 3, pp. 131–139, May–June, 1997. Original article submitted December 7, 1995; revision submitted March 25, 1996.

boundary conditions is sufficient to determine the kinematics of a medium. We shall substitute equalities (1.2) into (1.1) and obtain a closed system in terms of displacements. Clearly, the functionals R_{ij} do not exert an effect on the kinematics only if the strain and strain-rate distributions do not depend on the spatial coordinates. In addition, the material should be homogeneous, and the mass forces, including inertial ones, should be negligible. The homogeneity condition for strains leads to the following system of differential equations:

$$\frac{dx_i}{dt} = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3, \tag{1.5}$$

where the coefficients a_{ij} depend only on time t.

The solution of system (1.5) is representable in the form

$$x_i(t) = b_{i1}x_1^0 + b_{i2}x_2^0 + b_{i3}x_3^0. \tag{1.6}$$

The matrices $B = (b_{ij})$ and $A = (a_{ij})$ are interrelated: $A = (dB/dt)B^{-1}$.

It is easy to obtain conditions that are sufficient to realize the process (1.5). Let the deformed region be bounded by a closed surface S^0 at moment t^0. We shall specify on it the velocity vector according to equalities (1.5). Let the mass forces be absent, and the loading be of a quasi-static character. Inertial terms can hence be ignored, and there is no need for initial conditions. The solution is assumed to be unique. In particular, we exclude the rheological instability [1, 2], shear localization, fracture, etc. If equalities (1.5) are satisfied at the boundary, they are satisfied inside the region as well. In other words, with the velocities specified at the boundary, the kinematics of deformation inside the region is the same for elastic, viscous, elastoviscoplastic, and any other materials. Formally, one can say that equalities (1.5) yield a set of universal solutions of system (1.1)–(1.4) for any types of governing equations.

2. General Classification of Homogeneous Flows. We shall analyze system (1.5). There are no particular difficulties in the construction of its solutions. Moreover, from the very beginning one can proceed from equalities (1.6) which yield one or another specific flow for any choice of the matrix B. The problem lies in another direction. The set of all affine flows (1.6) depend on nine scalar functions of one argument. This is a fairly wide class. The problem is to narrow this class and to select from it the flows that could be realized in practice.

Let us go over to system (1.5). If one introduces, for the coefficients a_{ij}, the notation

$$\frac{dx_1}{dt} = \varepsilon_{11}x_1 + (\varepsilon_{12} - \Omega_3)x_2 + (\varepsilon_{13} + \Omega_2)x_3, \quad \frac{dx_2}{dt} = (\varepsilon_{12} + \Omega_3)x_1 + \varepsilon_{22}x_2 + (\varepsilon_{23} - \Omega_1)x_3,$$
$$\frac{dx_3}{dt} = (\varepsilon_{13} - \Omega_2)x_1 + (\varepsilon_{23} + \Omega_1)x_2 + \varepsilon_{33}x_3, \tag{2.1}$$

then one can say that all elements of the medium are subjected to a complex loading. Here $\Omega = \{\Omega_1, \Omega_2, \Omega_3\}$ is the velocity of the rotation vector, and ε_{ij} are, as before, the components of the strain-rate tensor.

It should be noted that, in this formulation, the superimposing of rotation is of a nontrivial character and affects considerably the kinematics as a whole. The role of rotation can be explained as follows. Let us assume that in the system of coordinates $Ox_1x_2x_3$, a loading device which is switched on for time Δt impacts a body by the strains $\varepsilon_{ij}\Delta t$ during this time interval. In addition, the principal axes of the strain-rate tensor are immobile (simple loading). Let us switch on the device for time Δt. Each point of the specimen undergoes the corresponding displacements. After that, we switch off the loading device and rotate, during the next time interval, the body as a rigid whole by an angle $|\Omega|\Delta t$ about the vector Ω. Again, the body receives the strains $\varepsilon_{ij}\Delta t$, etc. As a result, we will come to a complex loading with continuous rotations of the strain tensor's axes ($\Delta t \to 0$).

In [3], the author have studied a particular case where a simple shear was involved as a starting deformation (plane-parallel Couette flow). In [4], Ovsyannikov has constructed a class of exact solutions of the problem of the motion of an ideal fluid with a free boundary and a linear velocity field. Below, the general case (2.1) will be studied as a continuation of [3, 5].

Let us first confine ourselves to steady-state flows ($a_{ij} =$ const). As is known, the character of solution of linear systems depends on the eigenvalues of the matrix A. In a three-dimensional space, at least one of them is real. The particles that lie along their eigenvectors can move only along this vector: $dx/dt = Ax = \lambda x$, where $\lambda = \text{const}$ and x is the column (x_i). Therefore, there are systems of coordinates in which either $a_{13} = a_{23} = 0$ or $a_{31} = a_{32} = 0$ ($\varepsilon_{13} = \Omega_2$ and $\varepsilon_{23} = -\Omega_1$). We shall dwell on the second variant. Without loss of generality, one can assume that $\varepsilon_{12} = 0$. Then, we have

$$\dot{x}_1 = \varepsilon_{11}x_1 - \Omega_3 x_2 + 2\Omega_2 x_3, \quad \dot{x}_2 = \Omega_3 x_1 + \varepsilon_{22}x_2 - 2\Omega_1 x_3, \quad \dot{x}_3 = \varepsilon_{33}x_3. \tag{2.2}$$

The eigenvalues are $\lambda_{1,2} = \varepsilon/2 \pm (1/2)\sqrt{D}$, $\lambda_3 = \varepsilon_{33}$, where $D = (\varepsilon_{11} - \varepsilon_{22})^2 - 4\Omega_3^2$ and $\varepsilon = \varepsilon_{11} + \varepsilon_{22}$.

According to (2.2), the particles that are in the plane $x_3 = 0$ cannot leave this plane. In view of this, we shall first consider flow kinematics in this plane. Let us assume that at a certain moment all links between the particles in the medium break. And some effective mass forces begin to act as links. It is clear from (2.2) that, for $x_3 = 0$, the components of these forces have the form

$$\ddot{x}_1 = (\varepsilon_{11}^2 - \Omega_3^2)x_1 - \varepsilon\Omega_3 x_2, \quad \ddot{x}_2 = \varepsilon\Omega_3 x_1 + (\varepsilon_{22}^2 - \Omega_3^2)x_2. \tag{2.3}$$

The kinematics now can be represented as the motion of a set of material particles in the field of forces (2.2) with the initial velocities $\dot{x}_1 = \varepsilon_{11}x_1^0 - \Omega_3 x_2^0$ and $\dot{x}_2 = \Omega_3 x_1^0 + \varepsilon_{22}x_2^0$. One can readily see that the mass forces have a potential only for $\Omega_3\varepsilon = 0$. The case $\Omega_3 = 0$ corresponds to the biaxial tension (compression) along the fixed directions Ox_1 and Ox_2. To the case $\varepsilon = 0$ and $\Omega_3 \neq 0$ corresponds the central field of forces. The particle trajectories are either hyperbolas ($D > 0$) or ellipses ($D < 0$), or a set of parallel straight lines ($D = 0$). For $\varepsilon \neq 0$, the indicated trajectories are deformed because of all-around compression or tension. The field of forces then loses its property of potentiality, and, in general, the material points either move off to infinity or infinitely approach the center. If $\lambda_1 = 0$ or $\lambda_2 = 0$, then the straight line $x_2/x_1 = \varepsilon_{11}/\Omega_3$ is immobile.

It is easy to understand how this plane pattern unfolds in space. Let us consider the orientation of the eigenvector which corresponds to λ_3. We assume that $\lambda_3 \neq \lambda_{1,2}$. The eigenvector is then directed along the straight line

$$x_1 = 2\,\frac{\Omega_1\Omega_3 - \Omega_2(\varepsilon_{22} - \varepsilon_{33})}{(\varepsilon_{11} - \varepsilon_{33})(\varepsilon_{22} - \varepsilon_{33}) + \Omega_3^2}\,x_3, \quad x_2 = 2\,\frac{\Omega_2\Omega_3 + \Omega_1(\varepsilon_{11} - \varepsilon_{33})}{(\varepsilon_{11} - \varepsilon_{33})(\varepsilon_{22} - \varepsilon_{33}) + \Omega_3^2}\,x_3. \tag{2.4}$$

The latter equation in system (2.2) shows that the plane x_3 moves from the immobile plane $x_3 = 0$ according to the law $x_3(t) = x_3^0\exp(\varepsilon_{33}t)$. In this plane, the same pattern is observed as in the plane $x_3 = 0$. Here to the center $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ corresponds the point (2.4).

Special cases are implemented only if at least one of the values of $\lambda_{1,2}$ tends to λ_3. The straight line (2.3) approaches the plane $x_3 = 0$, and the transfer mechanism described above degenerates.

Thus, we can give the following general classification of homogeneous flows.

1. Discriminant $D < 0$. These are flows in which the material particles move around the center in elliptical trajectories; we call them *elliptical* flows.

2. $D > 0$, $\lambda_3 \neq \lambda_1$, and $\lambda_3 \neq \lambda_2$. These are flows in which the particles travel along hyperbolic trajectories; we call them *hyperbolic* flows.

3. $D = 0$. These are the remaining flows.

We shall consider homogeneous flows from the point of view of the possibility of their realization. In this case, another classification is more important. Precisely as for a "phase fluid" [6], the flow is called finite if the particle trajectories lie in a limited region and infinite in the opposite case. In practice, one deals only with bounded regions. In a stationary regime, infinite flows can therefore be implemented only under the condition that a new material inflows and outflows through the boundary of the region. The processes in which the deformed specimen consists of one and the same material points are more convenient. In a stationary regime, this is possible only for finite flows: $D < 0$, $\varepsilon_{33} \leqslant 0$, and $\varepsilon_{11} + \varepsilon_{22} \leqslant 0$. If a strict inequality is satisfied in at least one of the two above equations, the material undergoes compression which is virtually nonrealizable. For this reason, one should set $\varepsilon_{33} = 0$ and $\varepsilon_{11} + \varepsilon_{22} = 0$. After that, it is necessary to choose an appropriate shape for the specimen (i.e., the surface S_0), and to specify the velocity vector belonging to the class (1.5) at its boundary.

Let us consider a class of flows for which the surface S_0 goes over into itself. This implies that upon deformation the external configuration of the specimen remains unchanged. The idea of this step is associated with simulation of the Earth's tidal deformation [7]. In the model of tidal waves, precisely this situation was realized and proved to be appropriate for practical implementation.

The general description of affine flows of a similar type can be obtained as follows. Let us introduce a new system of coordinates $Oy_1y_2y_3$ and fix in it a body of revolution bounded by the surface S^*. We shall map S^* onto the coordinate system $Ox_1x_2x_3$ by the following affine transformation:

$$x_1 = k_1y_1, \quad x_2 = k_2y_2, \quad x_3 = k_3y_3. \tag{2.5}$$

Deformation (2.5) is called the starting deformation. As S_0, we choose an image of S^* in the coordinates $Ox_1x_2x_3$. We now force the preimage of S^* to rotate about the axis of symmetry at each moment of time and perform the transformation (2.5). Since the surface S^* goes over into itself, the surface S^0 preserves this property. Clearly, the internal points of the region S^0 undergoes the affine deformation. Let us determine its parameters. Let the rotation velocity $\boldsymbol{\omega} = \{\omega_1, \omega_2, \omega_3\}$ be constant. In the coordinate system Oy_i, we then have

$$\dot{y}_1 = -\omega_3y_2 + \omega_2y_3, \quad \dot{y}_2 = \omega_3y_1 - \omega_1y_3, \quad \dot{y}_3 = -\omega_2y_1 + \omega_1y_2. \tag{2.6}$$

Differentiating (2.5) with respect to time, using (2.6), and replacing y_i by x_ik_i, we obtain

$$\dot{x}_1 = -\frac{k_1\omega_3}{k_2}x_2 + \frac{k_1\omega_2}{k_3}x_3, \quad \dot{x}_2 = \frac{k_2\omega_3}{k_1}x_1 - \frac{k_2\omega_1}{k_3}x_3, \quad \dot{x}_3 = -\frac{k_3\omega_2}{k_1}x_1 + \frac{k_3\omega_1}{k_2}x_2. \tag{2.7}$$

Here the eigenvalues are $\lambda_{1,2} = \pm i\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$, $\lambda_3 = 0$. Hence, the class of flows (2.7) coincides with the above class of finite flows of the type $\varepsilon_{33} = 0$ and $\varepsilon_{11} + \varepsilon_{22} = 0$. [Evidently, the variables x_i in equalities (2.2) and (2.7) refer to different coordinate systems.] The representation of the flows (2.2) in the form of (2.7) allows one to use transformations (2.5) and (2.6) as an algorithm for choosing the shape of the specimen and the program of its loading. Let us now pass over to analysis of particular flows.

3. Plane Flows. Let us assume that the rotation is performed about the axis Ox_3, i.e., $\Omega_1 = \Omega_2 = 0$. The guide line (2.4) becomes vertical, and deformation on the whole becomes plane. We shall consider elliptic flows of the type $D < 0$, $\varepsilon_{11} + \varepsilon_{22} = 0$. As a preimage of the body, we select a right circular cylinder and subject it to biaxial compression in the directions orthogonal to the generatrix. As a result, the cylinder becomes elliptic but remains a right one. Rotation of the preimage about the axis of symmetry produces the plane elliptic flow:

$$\dot{x}_1 = kx_1 - \Omega_3x_2, \quad \dot{x}_2 = \Omega_3x_1 - kx_2 \qquad (k = \varepsilon_{11} = -\varepsilon_{22}), \tag{3.1}$$

to which corresponds the central field of effective mass forces possessing a potential. Therefore, the law of revolution of material particles around the center is Kepler's law:

$$\mathbf{vn} = 0; \qquad |\mathbf{v} \times \mathbf{r}| = \Omega_3(x_1^0)^2 = \text{const}. \tag{3.2}$$

Here $\mathbf{n}$ is the normal to the boundary, $\mathbf{r}$ is the radius-vector drawn from the origin of coordinates, and $\mathbf{v}$ is the velocity vector. For definiteness, we take a material point with the initial coordinates $(x_1^0, 0)$. The boundary of the region coincides with the trajectory of the point $(x_1^0, 0)$ and is an ellipse with semiaxes

$$a = x_1^0\sqrt{\frac{\Omega_3}{\Omega_3 - k}}, \qquad b = x_1^0\sqrt{\frac{\Omega_3}{\Omega_3 + k}}. \tag{3.3}$$

The ellipse's axes are directed along the bisectors $x_2 = \pm x_1$. Hence, if the Kepler's velocity distribution is specified at the boundary of the elliptic region, then only a uniform strain and stress distribution can exist inside the region. Moreover, the loading should be fairly slow to avoid inertial effects.

Any similar requirements can be met only approximately. The question therefore arises as to how the flow (3.1) changes if the inertial effects become pronounced. It is impossible here to obtain results which are not dependent on the rheology of the medium. Let us consider the case of a linearly viscous fluid. As usually, we shall introduce nondimensional variables keeping for them the notation of dimensional ones. We have derived equalities (3.2) as a corollary of (3.1). Now the formulation is different. It is necessary to treat equalities (3.2) and (3.3) as the prespecified boundary conditions. The boundary-value problem is formulated for stationary Navier–Stokes equations:

$$\Delta v_1 - \frac{\partial p}{\partial x_1} = \mathrm{Re}\left(v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2}\right),$$
$$\Delta v_2 - \frac{\partial p}{\partial x_2} = \mathrm{Re}\left(v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2}\right), \qquad \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0, \tag{3.4}$$

where p is the pressure and Re is the Reynolds number. The exact solution of problems (3.2)–(3.4) has the form

$$v_1 = kx_1 - \Omega_3 x_2, \quad v_2 = \Omega_3 x_1 - kx_2, \quad p = p^0 + \mathrm{Re}\,\frac{\Omega_3^2 - k^2}{2}\,(x_1^2 + x_2^2), \tag{3.5}$$

where p^0 is an additive constant. The form of the solution of (3.5) will not change if the boundary conditions (3.2) are set on the hyperbolas ($\Omega_3 < k$) or on a couple of parallel straight lines ($\Omega_3 = k$, Couette flow). It is clear from (3.5) that the constant-pressure curves are circles. The pressure increases with distance from the center for elliptical flows and decreases for hyperbolic ones, while, for the Couette flow, it remains constant. The solution also shows that the inertial forces are completely compensated by the pressure gradient. That is why the flow kinematics does not depend on inertial forces. This conclusion remains true for any Re values. However, an additional question arises concerning the stability of the flow (3.3) with increasing Re. It is known that, for plane-parallel Couette flow ($\Omega_3 \to k$ and $a \to \infty$), stability is preserved against any disturbances [8]. A similar result is likely to occur also for elliptic flows with the Kepler boundary conditions.

Thus, solution (3.5) illustrates a flow in which specifying the boundary conditions for velocities guarantees a definite flow kinematics which is not dependent on Re. Naturally, the question arises whether there are other flows with this property.

The problem can be formulated as follows: it is necessary to describe a class of flows satisfying system (3.4) with the field of velocities v_1 and v_2 independent of Re (of course, the pressure is Re-dependent). We differentiate the first two equations with respect to the parameter Re and use the conditions $\partial v_i/\partial \mathrm{Re} = 0$. As a result, we obtain two new equations

$$v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} = \frac{\partial Q}{\partial x_1}, \quad v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} = \frac{\partial Q}{\partial x_2}. \tag{3.6}$$

Here $Q = -\partial p/\partial \mathrm{Re}$. The opposite situation is easy to show: if the field of inertial forces has the potential (3.6), the flow kinematics is not dependent on Re. We introduce the stream function Φ:

$$v_1 = -\frac{\partial \Phi}{\partial x_2}, \qquad v_2 = \frac{\partial \Phi}{\partial x_1}.$$

Let us exclude the variables p and Q. As a result, we obtain the redeterminated system

$$\Delta\Delta\Phi = 0, \qquad \frac{\partial \Phi}{\partial x_1}\frac{\partial \Delta\Phi}{\partial x_2} - \frac{\partial \Phi}{\partial x_2}\frac{\partial \Delta\Phi}{\partial x_1} = 0. \tag{3.7}$$

The meaning of the equations is absolutely clear. The assumption of the nondependence of the kinematics on Re also incorporates the limiting cases where Re $\to 0$ and Re $\to \infty$. To the case Re $\to 0$ corresponds the first equation in (3.7), i.e., the creeping approximation, and to Re $\to \infty$ corresponds the second equation in (3.7), the properties of a viscous fluid becoming close to those of an ideal fluid. We introduce a complex variable $z = x_1 + ix_2$. The general solution of the first equation of (3.7) can then be represented as (Goursat's formula [9])

$$2\Phi = \bar{z}\varphi + \chi + z\bar{\varphi} + \bar{\chi}, \tag{3.8}$$

where $\varphi(z)$ and $\chi(z)$ are arbitrary analytical functions of argument z. Substitution of (3.8) into the second

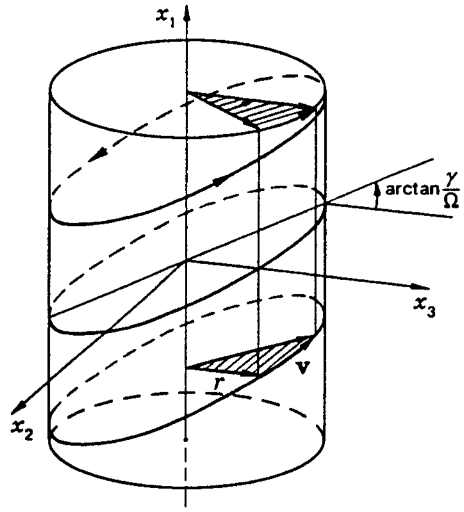

Fig. 1

equation of (3.7) yields the equation

$$\mathrm{Im}[\varphi''(\varphi + z\bar{\varphi}' + \bar{\psi})] = 0 \quad (\psi = \chi'). \tag{3.9}$$

Equality (3.9) gives a comprehensive description of a class of flows in which the kinematics is not dependent on Re. In particular, we can obtain a plane elliptic flow with the Kepler boundary conditions at the boundary if we set $\varphi''(z) = 0$ and $\psi(z) = kz$.

4. Spatial Flows. The search for spatial flows which are suitable for realization is more easy to perform by means of the algorithm (2.5)–(2.7). Let us choose a body shaped as a right circular cylinder. We set into correspondence to it another body which has the same shape but was subjected to shear strain. The shear was performed in such a way that the cylinder bases remained circular. We rotate the preimage about the axis of symmetry with a certain constant angular velocity θ. Deformation of the image can then be described by the relations

$$v_1 = \gamma x_2 + \theta x_3, \quad v_2 = 0, \quad v_3 = -\theta x_1, \quad \gamma,\, \theta = \mathrm{const}. \tag{4.1}$$

According to (4.1), the bases of an oblique cylinder rotate as rigid ones with the same constant angular velocity θ and so do all the body's cross sections parallel to the bases. In [5], the author described the realization of (4.1) and gave experimental data on the deformation of bulk material.

Next, we perform another type of shear of the preimage — along the axis of symmetry. Its lateral surface remains circular cylindrical, while the bases become ellipses (see Fig. 1). The rigid rotation of the preimage produces the following uniform deformation process:

$$v_1 = \gamma x_2, \quad v_2 = -\Omega x_3, \quad v_3 = \Omega x_2. \tag{4.2}$$

All trajectories of the material particles are ellipses which are formed in cutting a circular cylindrical surface by planes parallel to the bases. The law of particle revolution around elliptic orbits is the Kepler law: the particle radius vector covers equal areas for equal periods. The orbit plane constitutes a constant angle with the plane Ox_2x_3. Therefore, the projection of a material point onto the Ox_2x_3 plane moves in a circle with a constant angular velocity. This circumstance can be used in constructive implementation of the Kepler boundary conditions.

A class of loadings of the type of (2.7) exhibits one interesting property. Let us choose an S^0-shaped specimen in a natural free state and transform it into S^* according to the law (2.5). To this transformation correspond some invariants of the strain tensor. Subsequent loading should be performed according to the program (2.7). It is easy to show that this loading leads only to rotation of the principle axes of the strain tensor, while the values of the tensor invariants remain unchanged. Thus, in this case, a special type of neutral loading with continuous rotation of the strain tensor's axes occurs. In the case of (4.1), the first principal direction rotates uniformly in the plane Ox_1x_3, while two others describe conic surfaces so that the angle between each of them and the plane Ox_1x_3 remains constant. In the case of (4.2), a similar picture occurs relative to the plane Ox_2x_3.

We now consider the role of inertial effects for spatial flows. Let, as previously, the velocity distribution (2.1) be specified at the boundary of a certain region. The flow is assumed to be of a steady-state character, but inertial forces are not negligible. The question arises as to which additional conditions should be adopted for the strain and strain-rate distributions to remain, as before, homogeneous. Just as in the plane case, it is easy to show that the incompressibility condition of the medium is here a decisive one.

We assume that the constitutive equations of a medium depend only on the stress-tensor deviator. In other words, the additive hydrostatic pressure has no effect on the kinematics of the medium. For affine flows of similar media, the nonuniformity of the pressure can be assumed. In this case, the inertial forces should be completely compensated by the pressure gradient. The latter is equivalent to the existence of the potential of inertial forces.

We shall write relations of the form of (3.6) for the three-dimensional case, excluding the potential and replacing the velocities by relations (2.1). After simple manipulations, we obtain the following homogeneous system:

$$-(\varepsilon_{22}+\varepsilon_{33})\Omega_1+\varepsilon_{12}\Omega_2+\varepsilon_{13}\Omega_3=0, \qquad \varepsilon_{12}\Omega_1-(\varepsilon_{11}+\varepsilon_{33})\Omega_2+\varepsilon_{23}\Omega_3=0,$$
$$\varepsilon_{13}\Omega_1+\varepsilon_{23}\Omega_2-(\varepsilon_{11}+\varepsilon_{22})\Omega_3=0. \tag{4.3}$$

The nontrivial solutions of the system allows one to describe flows that preserve their uniform character not only under quasi-static loading, but also when inertial forces become pronounced. Let the vector of rotation $\boldsymbol{\Omega}\neq 0$. We direct it along the axis Ox_3. Then $\Omega_1=\Omega_2=0$. Relations (4.3) produce $\varepsilon_{13}=\varepsilon_{23}=0$ and $\varepsilon_{11}+\varepsilon_{22}=0$. From the incompressibility condition, it follows that $\varepsilon_{33}=0$ as well. This result implies that the flow should be plane, i.e., for $\boldsymbol{\Omega}\neq 0$, there are no spatial noninertial flows.

Let $\boldsymbol{\Omega}=0$. Equations (4.3) are then satisfied for any ε_{ij}. Thus, any homogeneous flow in which the principle axes of the strain-rate tensor do not rotate around material volumes is not dependent on inertial forces.

5. Nonstationary Flows. In many cases, it is necessary to study loadings with broken trajectories, cyclic shears, etc., which is possible only in nonstationary regimes. Formally, this means that, in Eqs. (2.1), it is necessary to admit the time dependence of the coefficients. The variety of unsteady-state flows is much richer than that of steady-state ones. The classification considered, however, allows one to describe also the most important unsteady-state flows. We shall consider the mappings (2.5) and (2.6). Let us rotate the preimage S^0 about the symmetry axes according to an arbitrary program, admitting a variable velocity and variable directions of rotation, etc. It is evident that, just as in the steady case, the surface S^* goes, as before, over to itself; correspondingly, the specimen surface S^0 also goes over to itself. The material inside the region undergoes an affine unsteady deformation. A general description of the flows of this class can be obtained if one uses a sphere as a preimage S^*. The sphere can be rotated around the center by any program. Here the surface of the sphere S^* and, hence, its image S^0 will go over to itself.

One can indicate another method of producing unsteady flows. Let (1.6) be the solution of some unsteady equations (1.5). The coefficients b_{ij} depend on the time t. We assume t to be not the time but the loading parameter which depends on the physical time τ: $t=t(\tau)$. No limitations are imposed on the monotone character of the function $t(\tau)$. This interpretation requires stricter conditions on the rate of quasi-static loading, but permits one to expand the class of homogeneous flows that are appropriate for implementation. In particular, in this formulation, one can use flows that are infinite relative to t. For example, a simple shear (Couette flow) is infinite relative to t. However, if we adopt the dependence $t=\sin\tau$, the shear relative to τ becomes infinite, and loading becomes cyclic.

In conclusion, we would like to note the following circumstance. The practical realization of any ideal requirements is always of an approximate character. It is impossible to avoid perturbations associated with the effect of inertial forces, the inaccuracy in the implementation of boundary conditions, etc. In some cases, such deviations give rise to new effects which are of independent interest [5, 7].

6. Conclusions.

1. The loading methods that ensure a uniform stress-strain distributions in a medium have been completely described. Such methods are ideal for setting up experiments to study the constitutive equations for various media.

2. We have indicated the loading methods that can be used for the study of complex rheological media and also for powdered and loose materials. The cases where inertial forces have no effect on the homogeneous character of flow have been considered.

This work was supported by the Russian Government and International Science Foundation (Grant No. NPD300).

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